# The Curvature and Geodesics of the Torus

http://www.rdrop.com/~half/math/torus/index.xhtml

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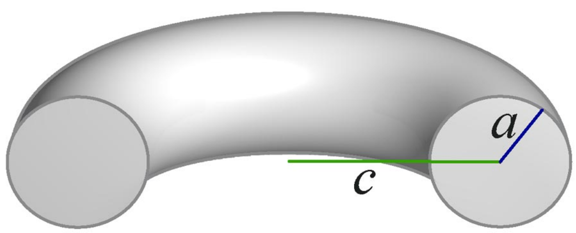
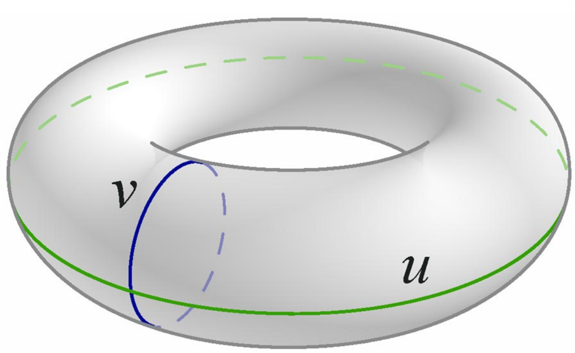
The torus is a standard example in introductory discussions of the curvature of surfaces. However, calculation of some measures of its curvature are hard to find in the literature. This paper offers full calculation of the torus's shape operator, Riemann tensor, and related tensorial objects. In addition, we examine the torus's geodesics by comparing a solution of the geodesic equation with results obtained from the Clairaut parameterization. Families of geodesics are classified. Open questions are considered. The connection form and parallel transport on the torus are investigated in an appendix.

## 1. The Line Element and Metric

Our model of a torus has major radius $c$ and minor radius $a$. We only consider the ring torus, for which $c>a$.




We use a $u,v$ coördinate system for which planes of constant $u$ pass through the torus's axis.




We parameterize the surface $\boldsymbol{x}$ by $\mathbf{x}(u,v) = \left\{ \begin{array}{l} x = (c + a\cos v)\cos u \\ y = (c + a\cos v)\sin u \\ z = a\sin v \end{array} \right\}.$

We begin by calculating the coefficients $E$, $F$, and $G$ of the first fundamental form.

$$\boldsymbol{x}_u = (-(c + a\cos v)\sin u, (c + a\cos v)\cos u, 0)$$
$$\boldsymbol{x}_v = (-a\cos u\sin v, -a\sin u\sin v, a\cos v)$$

$$\begin{aligned} E &= \boldsymbol{x}_u \cdot \boldsymbol{x}_u \\ &= (-(c + a\cos v)\sin u)^2 + ((c + a\cos v)\cos u)^2 + 0 \\ &= (c + a\cos v)^2 \end{aligned}$$

$$\begin{aligned} F &= \boldsymbol{x}_u \cdot \boldsymbol{x}_v \\ &= (-(c + a\cos v)\sin u)(-a\sin v\cos u) + ((c + a\cos v)\cos u)(-a\sin v\sin u) + (0)(a\cos v) \\ &= a\sin v\cos u\sin u(c + a\cos v) - a\sin v\cos u\sin u(c + a\cos v) \\ &= 0 \end{aligned}$$

$$\begin{aligned} G &= \boldsymbol{x}_v \cdot \boldsymbol{x}_v \\ &= (-a\sin v\cos u)^2 + (-a\sin v\sin u)^2 + (a\cos v)^2 \\ &= a^2\sin^2 v\cos^2 u + a^2\sin^2 v\sin^2 u + a^2\cos^2 v \\ &= a^2\sin^2 v + a^2\cos^2 v \\ &= a^2 \end{aligned}$$

This gives us the line element $ds^2 = (c + a\cos v)^2 du^2 + a^2 dv^2$ and metric:

$$g_{ij} = \begin{bmatrix} (c + a\cos v)^2 & 0 \\ 0 & a^2 \end{bmatrix},\ g^{ij} = \begin{bmatrix} \frac{1}{(c+a\cos v)^2} & 0 \\ 0 & \frac{1}{a^2} \end{bmatrix}.$$

For later computations we'll need the partial derivatives of the metric:

$$g_{ij,u} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \text{ and } g_{ij,v} = \begin{bmatrix} -2a\sin v(c + a\cos v) & 0 \\ 0 & 0 \end{bmatrix}$$

## 2. The Shape Operator

The normal to the surface is $N = (\cos u\cos v, \sin u\cos v, \sin v)$. Taking the partial derivatives of $N$ with respect to $u$ and $v$ gives the shape operator in those directions:

$$-S(\boldsymbol{x}_u) = N_u = (-\sin u\cos v, \cos u\cos v, 0)$$
$$-S(\boldsymbol{x}_v) = N_v = (-\cos u\sin v, -\sin u\sin v, \cos v)$$

Comparing these to $\boldsymbol{x}_u$ and $\boldsymbol{x}_v$, the partial derivatives of the parameterization $\boldsymbol{x}$, we find that they are multiples:

$$S(\boldsymbol{x}_u) = -\frac{\cos v}{c + a\cos v}\boldsymbol{x}_u$$
$$S(\boldsymbol{x}_v) = -\frac{1}{a}\boldsymbol{x}_v$$

The Gaussian curvature $K$ is the determinant of $S$, and the mean curvature $H$ is the trace of $S$.

$$K = \begin{vmatrix} -\frac{\cos v}{c+a\cos v} & 0 \\ 0 & -\frac{1}{a} \end{vmatrix} = \frac{\cos v}{a(c+a\cos v)}$$

$$H = \tfrac{1}{2}\left(-\tfrac{\cos v}{c+a\cos v} - \tfrac{1}{a}\right) = \tfrac{1}{2}\left(-\tfrac{a\cos v}{a(c+a\cos v)} - \tfrac{c+a\cos v}{a(c+a\cos v)}\right) = \tfrac{c+2a\cos v}{2a(c+a\cos v)}$$

# 3. The Curvature Tensor

**The Christoffel symbols of the second kind**

$$\begin{aligned}\Gamma^u_{uu} &= \frac{1}{2}[g^{uu}(g_{uu,u} + g_{uu,u} - g_{uu,u}) + g^{uv}(g_{vu,u} + g_{vu,u} - g_{uu,v})] \\ &= \frac{1}{2}[g^{uu}(0+0-0) + 0(g_{vu,u} + g_{vu,u} - g_{uu,v})] \\ &= 0\end{aligned}$$

$$\begin{aligned}\Gamma^u_{uv} &= \frac{1}{2}[g^{uu}(g_{uv,u} + g_{uu,v} - g_{uv,u}) + g^{uv}(g_{vv,u} + g_{vu,v} - g_{uv,v})] \\ &= \frac{1}{2}[g^{uu}(0 + g_{uu,v} - 0) + 0(g_{vv,u} + g_{vu,v} - g_{uv,v})] \\ &= \frac{1}{2}g^{uu}g_{uu,v} \\ &= \frac{1}{2}\left(\frac{-2a\sin v(c+a\cos v)}{(c+a\cos v)^2}\right) \\ &= -\frac{a\sin v}{(c+a\cos v)}\end{aligned}$$

$$\begin{aligned}\Gamma^u_{vu} &= \frac{1}{2}[g^{uu}(g_{uu,v} + g_{uv,u} - g_{vu,u}) + g^{uv}(g_{vu,v} + g_{vv,u} - g_{vu,v})] \\ &= \frac{1}{2}[g^{uu}(g_{uu,v} + 0 - 0) + 0(g_{vu,v} + g_{vv,u} - g_{vu,v})] = \frac{1}{2}g^{uu}g_{uu,v} = \Gamma^u_{uv} \\ &= -\frac{a\sin v}{(c+a\cos v)}\end{aligned}$$

$$\begin{aligned}\Gamma^u_{vv} &= \frac{1}{2}[g^{uu}(g_{uv,v} + g_{uv,v} - g_{vv,u}) + g^{uv}(g_{vv,v} + g_{vv,v} - g_{vv,v})] \\ &= \frac{1}{2}[g^{uu}(0+0-0) + 0(g_{vv,v} + g_{vv,v} - g_{vv,v})] \\ &= 0\end{aligned}$$

$$\Gamma^v_{uu} = \frac{1}{2}[0(g_{uu,u} + g_{uu,u} - g_{uu,u}) + g^{vv}(0 + 0 - g_{uu,v})]$$
$$= -\frac{1}{2} g^{vv} g_{uu,v}$$
$$= -\frac{1}{2}\frac{1}{a^2}(-2a \sin v(c + a\cos v))$$
$$= \frac{1}{a}\sin v(c + a\cos v)$$
$$\Gamma^v_{uv} = \frac{1}{2}[g^{vu}(g_{uv,u} + g_{uu,v} - g_{uv,u}) + g^{vv}(g_{vv,u} + g_{vu,v} - g_{uv,v})]$$
$$= \frac{1}{2}[0(g_{uv,u} + g_{uu,v} - g_{uv,u}) + g^{vv}(0 + 0 - 0)]$$
$$= 0$$
$$\Gamma^v_{vu} = \frac{1}{2}[g^{vu}(g_{uu,v} + g_{uv,u} - g_{vu,u}) + g^{vv}(g_{vu,v} + g_{vv,u} - g_{vu,v})]$$
$$= \frac{1}{2}[0(g_{uu,v} + g_{uv,u} - g_{vu,u}) + g^{vv}(0 + 0 - 0)]$$
$$= 0$$
$$\Gamma^v_{vv} = \frac{1}{2}[g^{vu}(g_{uv,v} + g_{uv,v} - g_{vv,u}) + g^{vv}(g_{vv,v} + g_{vv,v} - g_{vv,v})]$$
$$= \frac{1}{2}[0(g_{uv,v} + g_{uv,v} - g_{vv,u}) + g^{vv}(0 + 0 - 0)]$$
$$= 0$$

Partial derivatives of the nonzero Christoffel symbols:

$$\Gamma^u_{uv,v} = \Gamma^u_{vu,v} = -\left[(a\sin v)(-1)(c + a\cos v)^{-2}(-a\sin v) + (c + a\cos v)^{-1}(a\cos v)\right]$$
$$= -(a\sin v)^2(c + a\cos v)^{-2} - (c + a\cos v)^{-1}(a\cos v)$$
$$= -\frac{(a\sin v)^2}{(c + a\cos v)^2} - \frac{a\cos v}{(c + a\cos v)}$$
$$\Gamma^v_{uu,v} = \frac{1}{a}[\sin v(-a\sin v) + (c + a\cos v)\cos v] = \frac{1}{a}[c\cos v + a\cos^2 v - a\sin^2 v]$$

### The Riemann tensor

Throughout this section we use the identity $R^i{}_{jkl} = -R^i{}_{jlk}$.

$$R^u{}_{uuu} = \Gamma^u_{uu,u} - \Gamma^u_{uu,u} - \Gamma^u_{uu}\Gamma^u_{uu} - \Gamma^u_{vu}\Gamma^v_{uu} + \Gamma^u_{uu}\Gamma^u_{uu} + \Gamma^u_{vu}\Gamma^v_{uu}$$
$$= 0 - 0 - 0 - \Gamma^u_{vu} \times 0 + 0 + 0 = 0$$
$$R^u{}_{uuv} = -R^u{}_{uvu} = \Gamma^u_{uv,u} - \Gamma^u_{uu,v} - \Gamma^u_{uv}\Gamma^u_{uu} - \Gamma^u_{vv}\Gamma^v_{uu} + \Gamma^u_{uu}\Gamma^u_{uv} + \Gamma^u_{vu}\Gamma^v_{uv}$$
$$= 0 - 0 - 0 - 0 + 0 + 0 = 0$$
$$R^u{}_{uvv} = \Gamma^u_{uv,v} - \Gamma^u_{uv,v} - \Gamma^u_{uv}\Gamma^u_{uv} - \Gamma^u_{vv}\Gamma^v_{uv} + \Gamma^u_{uv}\Gamma^u_{uv} + \Gamma^u_{vv}\Gamma^v_{uv}$$
$$= \Gamma^u_{uv,v} - \Gamma^u_{uv,v} - (\Gamma^u_{uv})^2 - 0 + (\Gamma^u_{uv})^2 + 0 = 0$$

$$
\begin{aligned}
R^u{}_{vuu} &= \Gamma^u_{vu,u} - \Gamma^u_{vu,u} - \Gamma^u_{uu}\Gamma^u_{vu} - \Gamma^u_{vu}\Gamma^v_{vu} + \Gamma^u_{uu}\Gamma^u_{vu} + \Gamma^u_{vu}\Gamma^v_{vu} \\
&= 0 - 0 - 0 - 0 + 0 + 0 = 0
\end{aligned}
$$

$$
\begin{aligned}
R^u{}_{vuv} = -R^u{}_{vvu} &= \Gamma^u_{vv,u} - \Gamma^u_{vu,v} - \Gamma^u_{uv}\Gamma^u_{vu} - \Gamma^u_{vv}\Gamma^v_{vu} + \Gamma^u_{uu}\Gamma^u_{vv} + \Gamma^u_{vu}\Gamma^v_{vv} \\
&= 0 - \Gamma^u_{vu,v} - (\Gamma^u_{vu})^2 - 0 + 0 - 0 \\
&= -\left[-\left(\frac{a\sin v}{c + a\cos v}\right)^2 - \frac{a\cos v}{c + a\cos v}\right] - \left(\frac{a\sin v}{c + a\cos v}\right)^2 \\
&= \left(\frac{a\sin v}{c + a\cos v}\right)^2 + \frac{a\cos v}{c + a\cos v} - \left(\frac{a\sin v}{c + a\cos v}\right)^2 \\
&= \frac{a\cos v}{c + a\cos v}
\end{aligned}
$$

$$
\begin{aligned}
R^u{}_{vvv} &= \Gamma^u_{vv,v} - \Gamma^u_{vv,v} - \Gamma^u_{uv}\Gamma^u_{vv} - \Gamma^u_{vv}\Gamma^v_{vv} + \Gamma^u_{uv}\Gamma^u_{vv} + \Gamma^u_{vv}\Gamma^v_{vv} \\
&= 0 - 0 - 0 - 0 + 0 + 0 = 0
\end{aligned}
$$

$$
\begin{aligned}
R^v{}_{uuu} &= \Gamma^v_{uu,u} - \Gamma^v_{uu,u} - \Gamma^v_{uu}\Gamma^u_{uu} - \Gamma^v_{vu}\Gamma^v_{uu} + \Gamma^v_{uu}\Gamma^u_{uu} + \Gamma^v_{vu}\Gamma^v_{uu} \\
&= 0 - 0 - 0 - 0 + 0 + 0 = 0
\end{aligned}
$$

$$
\begin{aligned}
R^v{}_{uuv} = -R^v{}_{uvu} &= \Gamma^v_{uv,u} - \Gamma^v_{uu,v} - \Gamma^v_{uv}\Gamma^u_{uu} - \Gamma^v_{vv}\Gamma^v_{uu} + \Gamma^v_{uu}\Gamma^u_{uv} + \Gamma^v_{vu}\Gamma^v_{uv} \\
&= 0 - \Gamma^v_{uu,v} - 0 - 0 + \Gamma^v_{uu}\Gamma^u_{uv} + 0 \\
&= -\frac{1}{a}(\sin v(-a\sin v) + \cos v(c + a\cos v)) + \frac{1}{a}\sin v(c + a\cos v)\left(-\frac{a\sin v}{c + a\cos v}\right) \\
&= -\frac{1}{a}[-a\sin^2 v + \cos v(c + a\cos v) + a\sin^2 v] \\
&= -\frac{1}{a}\cos v(c + a\cos v)
\end{aligned}
$$

$$
\begin{aligned}
R^v{}_{uvv} &= \Gamma^v_{uv,v} - \Gamma^v_{uv,v} - \Gamma^v_{uv}\Gamma^u_{uv} - \Gamma^v_{vv}\Gamma^v_{uv} + \Gamma^v_{uv}\Gamma^u_{uv} + \Gamma^v_{vv}\Gamma^v_{uv} \\
&= 0 - 0 - 0 - 0 + 0 + 0 = 0
\end{aligned}
$$

$$
\begin{aligned}
R^v{}_{vuu} &= \Gamma^v_{vu,u} - \Gamma^v_{vu,u} - \Gamma^v_{uu}\Gamma^u_{vu} - \Gamma^v_{vu}\Gamma^v_{vu} + \Gamma^v_{uu}\Gamma^u_{vu} + \Gamma^v_{vu}\Gamma^v_{vu} \\
&= 0 - 0 - \Gamma^v_{uu}\Gamma^u_{vu} - 0 + \Gamma^v_{uu}\Gamma^u_{vu} + 0 = 0
\end{aligned}
$$

$$
\begin{aligned}
R^v{}_{vuv} = -R^v{}_{vvu} &= \Gamma^v_{vv,u} - \Gamma^v_{vu,v} - \Gamma^v_{uv}\Gamma^u_{vu} - \Gamma^v_{vv}\Gamma^v_{vu} + \Gamma^v_{uu}\Gamma^u_{vv} + \Gamma^v_{vu}\Gamma^v_{vv} \\
&= 0 - 0 - 0 - 0 + 0 + 0 = 0
\end{aligned}
$$

$$
\begin{aligned}
R^v{}_{vvv} &= \Gamma^v_{vv,v} - \Gamma^v_{vv,v} - \Gamma^v_{uv}\Gamma^u_{vv} - \Gamma^v_{vv}\Gamma^v_{vv} + \Gamma^v_{uv}\Gamma^u_{vv} + \Gamma^v_{vv}\Gamma^v_{vv} \\
&= 0 - 0 - 0 - 0 + 0 + 0 = 0
\end{aligned}
$$

## The Ricci tensor

$$R_{ij} = R^m{}_{imj}$$

$$R_{uu} = R^m{}_{umu} = \frac{1}{a}\cos v(c + a\cos v)$$

$$R_{vv} = R^m{}_{vmv} = \frac{a\cos v}{c + a\cos v}$$

$$R_{ij} = \begin{bmatrix} \frac{1}{a}\cos v(c + a\cos v) & 0 \\ 0 & \frac{a\cos v}{(c + a\cos v)} \end{bmatrix}$$

**The Ricci scalar, a.k.a. the curvature scalar**

$$
\begin{aligned}
R &= g^{ij}R_{ij} = g^{uu}R_{uu} + g^{vv}R_{vv} \\
&= \left(\frac{1}{(c+a\cos v)^2}\right)\left(\frac{1}{a}\cos v(c+a\cos v)\right) + \left(\frac{1}{a^2}\right)\left(\frac{a\cos v}{c+a\cos v}\right) \\
&= \frac{\cos v}{a(c+a\cos v)} + \frac{\cos v}{a(c+a\cos v)} \\
R &= \frac{2\cos v}{a(c+a\cos v)}
\end{aligned}
$$

$R$ is twice the Gaussian curvature, as expected.

# 4. The Geodesic Equation

Let's look at the geodesic equation $\ddot{x}^a + \Gamma^a_{bc}\dot{x}^b\dot{x}^c = 0$. Plugging in Christoffel symbols and components of the Riemann tensor yields two equations.

$$
\begin{aligned}
&(i) \quad \ddot{u} + 2\Gamma^u_{uv}\dot{u}\dot{v} = \ddot{u} - \tfrac{2a\sin v}{c+a\cos v}\dot{u}\dot{v} = 0 \\
&(ii) \quad \ddot{v} + \Gamma^v_{uu}\dot{u}^2 = \ddot{v} + \tfrac{1}{a}\sin v(c+a\cos v)\dot{u}^2 = 0
\end{aligned}
$$

To solve (i), let $w = c + a\cos v$. Divide (i) by $\dot{u}$ and integrate:

$$
\begin{aligned}
\dot{w} &= -a(\sin v)\dot{v} \\
-2\dot{w} &= 2a(\sin v)\dot{v} \\
\int \frac{1}{\dot{u}}\ddot{u} &= \int -\frac{2}{w}\dot{w} \\
\ln \dot{u} &= -2\ln w + \ln k \\
&= \ln w^{-2} + \ln k \\
&= \ln(kw^{-2}) \\
\dot{u} &= kw^{-2} \\
&= \frac{k}{w^2} \\
&= \frac{k}{(c+a\cos v)^2}
\end{aligned}
$$

To solve (ii) multiply by $\dot{v}$ and integrate, using the same $w = c + a\cos v$ substitution:

$$\ddot{v}\dot{v} + \frac{1}{a}\sin v(c + a\cos v)(\dot{u})^2\dot{v} = 0$$

$$\ddot{v}\dot{v} + \frac{1}{a}\sin v(c + a\cos v)\frac{k^2}{(c + a\cos v)^4}\dot{v} = 0$$

$$\ddot{v}\dot{v} + \frac{k^2}{a}\frac{1}{(c + a\cos v)^3}\sin(v)\dot{v} = 0$$

$$(\sin v)\dot{v} = -\frac{1}{a}\dot{w}$$

$$\ddot{v}\dot{v} + \frac{k^2}{a}\frac{1}{w^3}\left(-\frac{1}{a}\right)\dot{w} = 0$$

$$\ddot{v}\dot{v} = \frac{k^2}{a^2}\frac{1}{w^3}\dot{w}$$

$$\int \ddot{v}\dot{v} = \frac{k^2}{a^2}\int \frac{1}{w^3}\dot{w}$$

$$\frac{1}{2}\dot{v}^2 = -\frac{k^2}{2a^2(c + a\cos v)^2} + \frac{1}{2}l$$

$$\dot{v}^2 = -\frac{k^2}{a^2(c + a\cos v)^2} + l$$

Which yields

$$\boxed{\begin{aligned} \dot{u} &= \frac{k}{(c + a\cos v)^2} \\ \dot{v} &= \pm\sqrt{-\frac{k^2}{a^2(c + a\cos v)^2} + l} \end{aligned}}$$

This is the general solution to the geodesic equation. To find actual geodesics, we must find a unit speed parameterization of the curve defined by $\dot{u}$, $\dot{v}$.

But first we check the solution. For convenience in the checks, we compute $\ddot{u}$ and $\ddot{v}$:

$$\ddot{u} = \frac{2ka\sin v}{(c + a\cos v)^3}\dot{v}$$

$$\dot{v} = \left(l - \frac{k^2}{a^2}(c + a\cos v)^{-2}\right)^{\frac{1}{2}}$$

$$\ddot{v} = \frac{1}{2}\left(l - \frac{k^2}{a^2}(c + a\cos v)^{-2}\right)^{-\frac{1}{2}}\left(-\frac{k^2}{a^2}(-2)(c + a\cos v)^{-3}\right)(-a\sin v)\dot{v}$$

$$\ddot{v} = \frac{1}{2}\left[\left(l - \frac{k^2}{a^2}(c + a\cos v)^{-2}\right)^{-\frac{1}{2}}\left(l - \frac{k^2}{a^2}(c + a\cos v)^{-2}\right)^{\frac{1}{2}}\right]\left(-\frac{k^2}{a^2}(-2)(c + a\cos v)^{-3}\right)(-a\sin v)$$

$$\ddot{v} = \frac{1}{2}\left(-\frac{k^2}{a^2}(-2)(c + a\cos v)^{-3}\right)(-a\sin v)$$

$$\ddot{v} = -\frac{k^2 \sin v}{a(c + a\cos v)^3}$$

i. Check of (*i*):

$$\frac{2ka\sin v}{(c + a\cos v)^3}\dot{v} - \frac{2a\sin v}{(c + a\cos v)}\frac{k}{(c + a\cos v)^2}\dot{v} =$$

$$\frac{2ka\sin v}{(c + a\cos v)^3}\dot{v} - \frac{2ka\sin v}{(c + a\cos v)^3}\dot{v} = 0 \ \checkmark$$

ii. Check of (*ii*)

$$-\frac{k^2 \sin v}{a(c + a\cos v)^3} + \frac{\sin v(c + a\cos v)}{a}\left(\frac{k}{(c + a\cos v)^2}\right)^2 =$$

$$-\frac{k^2 \sin v}{a(c + a\cos v)^3} + \frac{k^2 \sin v(c + a\cos v)}{a(c + a\cos v)^4} =$$

$$-\frac{k^2 \sin v}{a(c + a\cos v)^3} + \frac{k^2 \sin v}{a(c + a\cos v)^3} = 0 \ \checkmark$$

### The unclear geometric role of *k* and *l*

One problem with this solution to the geodesic equation is that we have two constants of integration, *k* and *l*, yet given a point on a surface a geodesic's path is determined by only one extra parameter, its direction. It's unclear from this solution precisely how *k* and *l* encode this information. This makes the solution to the geodesic equation useless for determining the paths of geodesics.

### No unit speed parameterization

For a curve $\alpha$ to be a geodesic, it must be a unit speed curve ($\|\dot{\alpha}(t)\| = 1$). Unfortunately, that pesky constant of integration *l* makes a general solution to this problem difficult.

$$\begin{aligned}\|\dot{\alpha}(t)\| &= \sqrt{\dot{u}^2+\dot{v}^2} \\ &= \sqrt{\frac{k^2}{(c+a\cos v(t))^4} - \frac{k^2}{a^2(c+a\cos v(t))^2} + l}\end{aligned}$$

If $l$ were $\frac{k^2}{4a^4}$, we could complete the square under the radical and integrate. However, we can't make that assumption. The meridian geodesics defined by $k=0$, $l=1$ are a counterexample.

Since we're not making much headway here, let's see whether the Clairaut parameterization helps.

## 5. The Clairaut Parameterization

Unfortunately, the parameterization we initially chose is the reverse of what is normally used for a Clairaut parameterization, so in this section the roles of $E$ and $G$ are reversed.

Recall our parameterization of the torus: $\begin{aligned} x &= (c+a\cos v)\cos u \\ y &= (c+a\cos v)\sin u \\ z &= a\sin v \end{aligned}$, and the first fundamental form $E=(c+a\cos v)^2$, $F=0$, $G=a^2$.

From O'Neill §7.5.5, a geodesic $\alpha$ can be parameterized as $\beta(v)=\boldsymbol{x}(u(v),v)$ where

$$\boxed{\begin{aligned}\frac{du}{dv} &= \pm\frac{h\sqrt{G}}{\sqrt{E}\,\sqrt{E-h^2}} \\ &= \pm\frac{ah}{(c+a\cos v)\sqrt{(c+a\cos v)^2-h^2}}\end{aligned}}$$

If we could integrate this, we'd have a nice formula for $u$ in terms of $v$:

$$u(v)=u(v_0)\pm ah\int_{v_0}^{v}\frac{dv}{(c+a\cos v)\sqrt{(c+a\cos v)^2-h^2}}$$

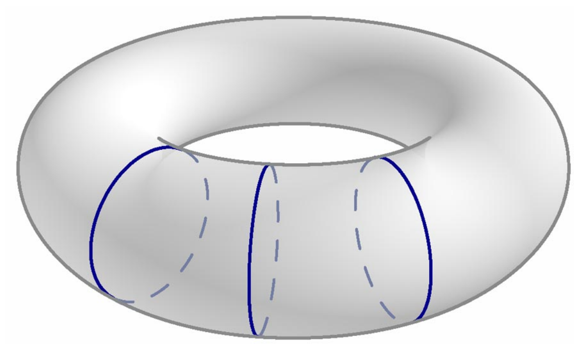
Alas, this integral likely has no closed form solution. But the formula for $\frac{du}{dv}$ is nice: it depends on only one parameter, $h$, the geodesic's slant. Following O'Neill §7.5.3, if $\alpha=\boldsymbol{x}(a_1,a_2)$ is a unit-speed geodesic and $\varphi$ the angle from $\boldsymbol{x}_v$ to $\alpha'$, there is a constant $h$ such that

$$\begin{aligned}h &= E(a_1)a_2' \\ &= \sqrt{E}\,(a_1)\sin\varphi \\ &= (c+a\cos a_1)\sin\varphi\end{aligned}$$

Hence $h$ measures of the angle between the geodesic and the $x_v$ curves.

The possible values of $h$ gives us an idea of the different kinds of geodesics that exist on the torus. The term under the radical must be real, hence $(c + a\cos v)^2 \geq h^2 \rightarrow |h| \leq c + a$. This allows us to classify the possible geodesics into several families. (We'll only consider positive values of $h$; negative values yield mirror image geodesics.) Note that technically we're considering *pregeodesics* here: to make them true geodesics, we'd need to find unit-speed parameterizations.

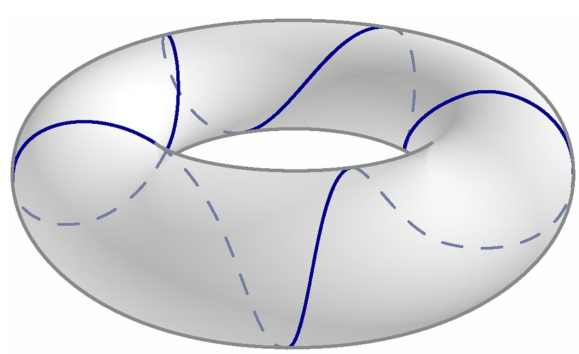
$\boldsymbol{h} = \boldsymbol{0} \rightarrow \frac{du}{dv} = 0$. These are the meridians:



An intuitive way to see that meridians are geodesics is to realize that the torus has a mirror symmetry through meridians. Anything that would push the geodesic off a meridian in one direction is balanced on the opposite side, so a geodesic that starts on a meridian cannot leave it.

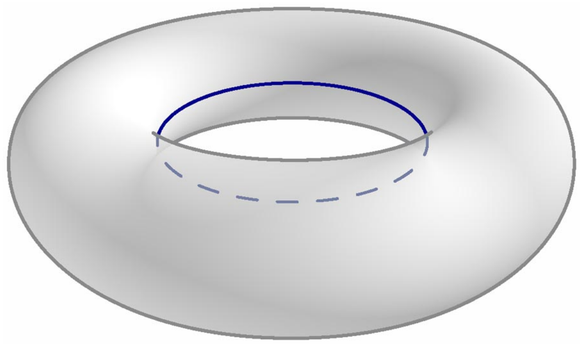
A similar argument can be made for both the inner and outer equators, which means they must be geodesics as well.

$\boldsymbol{0} < |h| < \boldsymbol{c} - \boldsymbol{a}$. These geodesics cross both the inner and outer equators. We call these geodesics *unbounded*, because they can pass through all points on the surface.

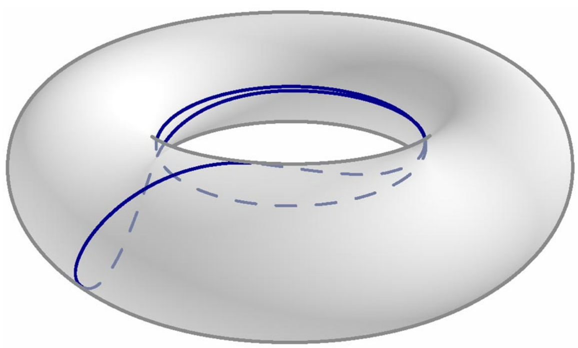


A consequence of the Clairaut relation $h = (c + a\cos a_1)\sin\varphi$ is that these geodesics cross the inner and outer equators at different angles. Note how the slant of the illustrated geodesic varies with $v$. A second consequence is that as $h$ increases, geodesics will approach tangency to the inner equator faster than to outer equator.

***h* = *c* − *a*.** As $h$ approaches this value from below, the angle a geodesic makes with the inner equator approaches zero. Hence when $h = c - a$, one geodesic is the inner equator.



What of the geodesics with $h = c - a$ which pass through other points on the torus? They're similar to the unbounded geodesics, but are asymptotic to the inner equator. (Our diagrams don't have enough resolution to show that these geodesics circle the inner equator endlessly without touching it.)



These "asymptotic" geodesics are an edge case of the next family of geodesics. This geodesic is unique barring rotation about the $z$ axis and reflection through the $xy$ plane.

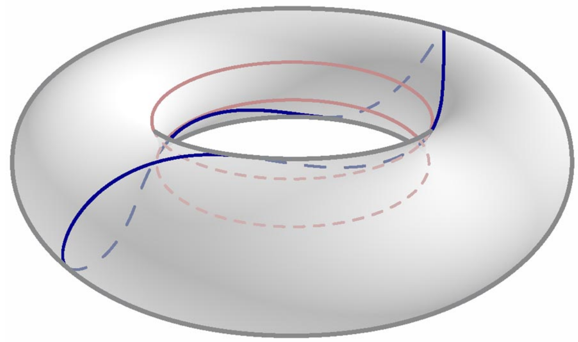
**$c - a < |h| < c + a$.** Another consequence of the Clairaut relation is that a geodesic $\alpha$ cannot leave the region $E \geq h^2$. For unbounded geodesics this restriction has no impact, but it does when $c - a < |h| < c + a$. If $\varphi = \frac{\pi}{2}$ for some $v_0$, $\alpha$ is tangent to the $v_0$ parallel. In that case,

$$(c + a\cos v)^2 \geq (c + a\cos v_0)^2$$

$$c + a\cos v \geq c + a\cos v_0$$

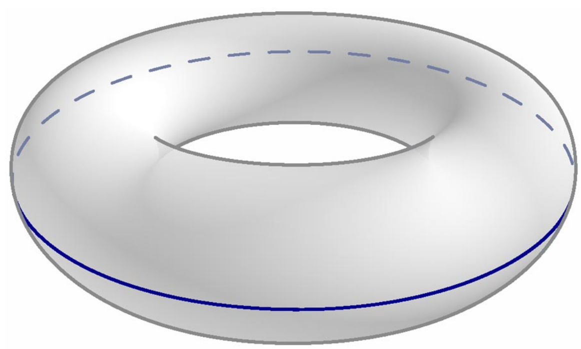
$$\cos v \geq \cos v_0$$

i.e., $\alpha$ is confined to the outer part of the torus between the $v_0$ and $-v_0$ parallels (the geodesic's *barrier curves*). We call these geodesics *bounded*. Here is one of the simplest bounded geodesics, which touches each barrier curve once:



As $v_0 \to \pi$, the region between the barrier curves grows to encompass the entire torus.

$\boldsymbol{h = c + a}$**.** As $h$ approaches $c + a$, the barrier curves approach the outer equator. Hence the one geodesic for which $h = c + a$ is the outer equator.



**Summary**

We can characterize all geodesics in terms of the absolute value of their slant $h$:

| $\lvert h \rvert$ | **Geodesics** |
|---|---|
| 0 | Meridians |
| $0 < \lvert h \rvert < c - a$ | Alternately cross both equators ("unbounded" geodesics) |
| $c - a$ | The inner equator, and geodesics asymptotic to it |
| $c - a < \lvert h \rvert < c + a$ | Cross outer equator but not inner equator (“bounded” geodesics) |
| $c + a$ | The outer equator |

For $\lvert h \rvert > c + a$, there are no real solutions to the geodesic equation.

# 6. The Clairaut Parameterization and the Geodesic Equation

**The mystery of *k* and *l* solved**

Returning to the question of the meaning of the constants of integration $k$ and $l$ which came out of the geodesic equation, we find that the formulation for $\frac{du}{dv}$ that comes from the Clairaut parameterization offers an answer.

$$\dot{u} = \frac{k}{(c + a\cos v)^2}$$

$$\dot{v} = \pm\sqrt{-\frac{k^2}{a^2(c + a\cos v)^2} + l}$$

(from the geodesic equation)

$$\dot{v} = \pm\sqrt{\frac{-k^2}{a^2(c + a\cos v)^2} + \frac{la^2(c + a\cos v)^2}{a^2(c + a\cos v)^2}}$$

$$\dot{v} = \pm\sqrt{\frac{-k^2 + la^2(c + a\cos v)^2}{a^2(c + a\cos v)^2}}$$

$$\dot{v} = \pm\frac{\sqrt{la^2(c + a\cos v)^2 - k^2}}{a(c + a\cos v)}$$

$$\frac{\dot{u}}{\dot{v}} = \pm\frac{k}{(c + a\cos v)^2}\frac{a(c + a\cos v)}{\sqrt{la^2(c + a\cos v)^2 - k^2}}$$

$$\frac{\dot{u}}{\dot{v}} = \pm\frac{ak}{(c + a\cos v)\sqrt{la^2(c + a\cos v)^2 - k^2}}$$

$\frac{\dot{u}}{\dot{v}}$ is identical to the Clairaut parameterization-derived formula for $\frac{du}{dv}$ when $k = h$ and $l = \frac{1}{a^2}$.

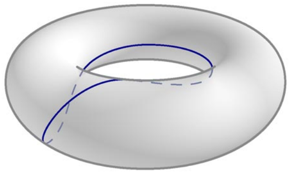
The two approaches are complementary. The formulas for $\dot{u}, \dot{v}$ derived from the geodesic equation can be used to compute geodesics that are singular in the formula derived from the Clairaut parameterization. In particular, the Clairaut parameterization-derived formula can’t be used to compute the inner and outer equator geodesics, but the formulas derived from the geodesic equation can.

# 7. A Gallery of Geodesics

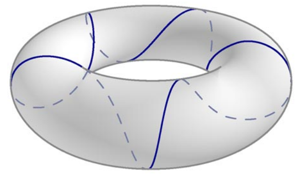
The majority of geodesics on the torus are not æsthetically pleasing. They are aperiodic and cover either the entire surface (if the geodesic is unbounded) or the outer region of the surface bounded by the barrier curves (if bounded). The rare exceptions are the geodesics which return to their starting point after just a few circuits around the $z$ axis.

Define the period of a geodesic as the number of circuits it makes around the $z$ axis before returning to its starting point. Most geodesics never return to their starting point, eventually covering either the entire torus surface or the region between barrier curves. However, there are geodesics that are pleasing; these are the unbounded geodesics of period 1, and the bounded geodesics of period 1 or 2.

The unbounded geodesics with period 1 cross each equator $n$ times ($n \geq 1$).

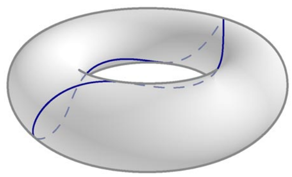


$n = 1$



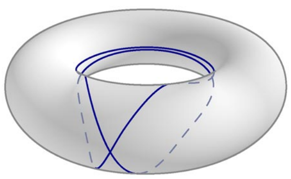
$n = 5$

The interesting bounded geodesics fall into two groups. Those of period 1 do not self-intersect. (For bounded geodesics, $n$ denotes how many times the geodesic touches each barrier curve.)

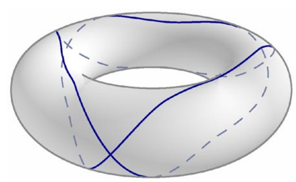


$n = 1$

The geodesics of period 2 intersect themselves $n$ times ($n$ odd, of course).



$n = 1$



$n = 3$

For bounded geodesics the allowed values of $n$ depend on the ratio $c/a$. Unbounded geodesics are not affected by $c/a$.

## 8. Open Questions

**The influence of $c/a$ on bounded geodesics**

The kinds of bounded geodesics one can find on a particular torus are determined not only by $h$, but also by the ratio $c/a$. For instance, given a torus with $\frac{c}{a} = \frac{3}{1}$, there is no period 1 bounded geodesic which touches each barrier curve more than once. Yet for a torus with $\frac{c}{a} = \frac{8}{1}$, there is a period 1 bounded geodesic which touches each barrier curve twice, and another which touches each barrier curve three times.

This raises a question: what is the range of $c/a$ of the toruses that contain bounded period 1 geodesics which touch each barrier curve exactly $n$ times, as a function of $n$? How about for different periods? There is no analytic apparatus I know of with which we can approach the problem. Calculation appears to be the only way to go.

Note that this restriction appears to apply only to bounded geodesics. There is no corresponding restriction for unbounded geodesics; by choosing an appropriate $h$, one can find a period 1 geodesic which crosses both equators as often as one pleases.

### Questions about critical values of *h*

Another open question concerns the values of $h$ which yield crowd-pleasing geodesics with periods 1 and 2. As $c/a$ changes, so does the value of $h$ which yields a particular pleasing geodesic (say, a period 1 geodesic which crosses both equators three times). Is there a simple relation between these two quantities?

A similar question exists for values of $h$ for a given $c/a$. Define $h_p$ as the value of $h$ which yields a period 1 geodesic which crosses both equators $p$ times. As $p$ increases, at what rate does $h_p$ converge to 0? Is this governed by a simple rule? What about period 2 bounded geodesics?

## 9. Numerically Calculating Geodesic Paths

Numerically calculating points on these geodesics is a little tricky; for bounded geodesics, $\frac{du}{dv}$ is undefined where the geodesic touches a barrier curve. After initial experiments with a spreadsheet, a short Perl script was used to generate points. A second script used successive approximation to find $h$ values which yielded geodesics with integral periodicity. The images in this paper are as accurate as I could make them, but there was a slight tradeoff between precise numeric solutions and illustrative power. Take the images with a grain of salt.

On a puzzling note, the second script worked extremely well for unbounded geodesics, but was less successful for bounded geodesics. The reason for the discrepancy is not clear.

## 10. Lessons Learned

The search for the geodesics of the torus led to the creation of Irons' First Law of Examples: *if an example seems obvious but you can't find it in the literature, it's more complex than you expect.*

## 11. References

Barrett O'Neill, *Elementary Differential Geometry, 2nd Ed.*, Harcourt, 1997; http://www.math.ucla.edu/~bon/.

*Thanks are due to professors Tevian Dray, for his encouragement, and Juha Pohjanpelto, for a fruitful suggestion.*

# Appendix A: Parallel Transport on the Torus

Our surface parameterization is an orthogonal patch ($F = \boldsymbol{x}_u \cdot \boldsymbol{x}_v = 0$), so we can easily compute the associated frame field $E_1$, $E_2$.

$$E_1 = \frac{\boldsymbol{x}_u}{\sqrt{E}} = (-\sin u, \cos u, 0)$$
$$E_2 = \frac{\boldsymbol{x}_v}{\sqrt{G}} = (-\cos u \sin v, -\sin u \sin v, \cos v)$$

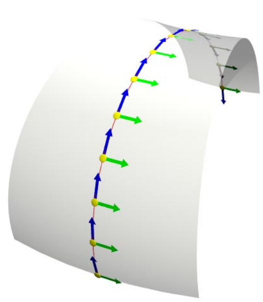
Let's check that their dot product is zero and their cross product is the normal $N = (\cos u \cos v, \sin u \cos v, \sin v)$.

$$\begin{aligned}
E_1 \cdot E_2 &= \sin u \cos u \sin v - \sin u \cos u \sin v + 0 \\
&= 0 \ \checkmark \\
E_1 \times E_2 &= \begin{vmatrix} \vec{x} & \vec{y} & \vec{z} \\ -\sin u & \cos u & 0 \\ -\cos u \sin v & -\sin u \sin v & \cos v \end{vmatrix} \\
&= (\cos u \cos v, \sin u \cos v, \sin^2 u \sin v + \cos^2 u \sin v) \\
&= (\cos u \cos v, \sin u \cos v, \sin v) \\
&= N \ \checkmark
\end{aligned}$$

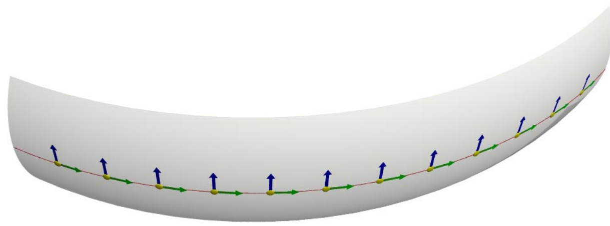
We can use partial derivatives of $E$ and $G$ to compute the connection form $\omega_{12}$, which encodes pretty much everything we'd ever want to know about how vectors change when parallel transported on the torus:

$$\omega_{12} = \frac{(\sqrt{E})_v}{\sqrt{G}} du + \frac{(\sqrt{G})_u}{\sqrt{E}} dv = \frac{a \sin v}{a} du + 0 = \sin v \, du$$

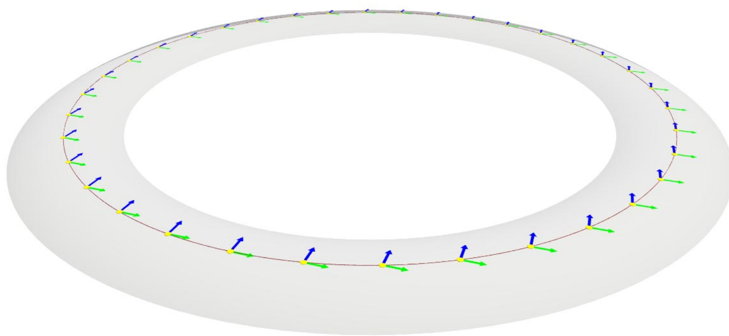
The *du* term tells us that parallel transport along lines of constant *u* (longitude lines) doesn't affect vectors. We could have predicted this from the symmetry of the torus; along a line of longitude, the neighborhoods to the left and right are mirror images, so there's no preferred direction for a vector to rotate.



The $\sin v$ term tells us that parallel transport along lines of constant $v$ (latitude lines) causes vectors to rotate through $2\pi \sin v$ during their journey back to their starting point. Let's look at some specific cases, starting with parallel transport along the outer equator. Here $\sin v = 0$, which means the vectors do not rotate as they are parallel transported along the outer equator.

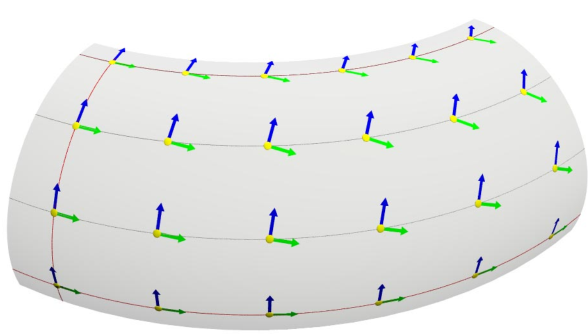


At the top of the torus ($v = \frac{\pi}{2}$) $\sin v = 1$, so a vector rotates through a full $2\pi$ during its journey. Note the angle between the blue vector and its path (red) as the vector is parallel transported.

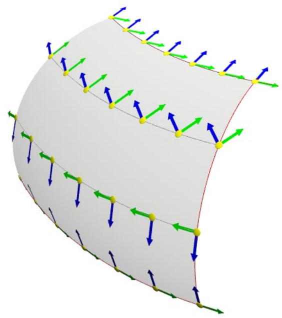


Something else interesting is happening simultaneously: the vector's origin is also rotating through$2\pi$. These rotations cancel, leaving the blue vector pointing in the same direction in the embedding space. Someone living on the torus would say the vector rotates as it is parallel transported, while someone living outside the surface would not. (At $v = \frac{\pi}{2}$ the torus's Gaussian curvature is zero, so it's not surprising that vectors parallel transported along that path don't appear to rotate in the embedding space.)

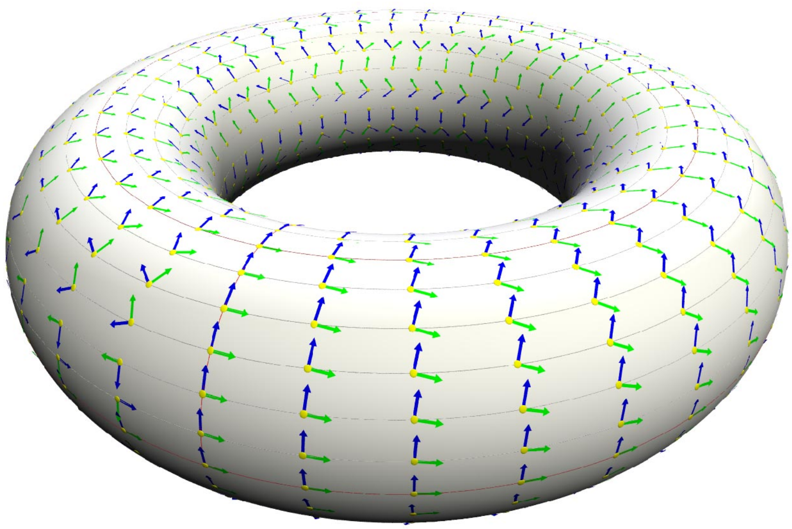
Parallel transport along other lines of latitude causes vectors to rotate varying amounts ($2\pi \sin v$). In the next illustration, four frames aligned with the $u$ and $v$ axes (at $v = 0$, $\frac{\pi}{6}$, $\frac{\pi}{3}$, and $\frac{\pi}{2}$) are parallel transported widdershins around the torus.



The second line of latitude from the bottom is at $v = \frac{\pi}{6}$, so vectors parallel transported along it will have rotated through $2\pi \sin\frac{\pi}{6} = 2\pi\frac{1}{2} = \pi$, as indeed they have.

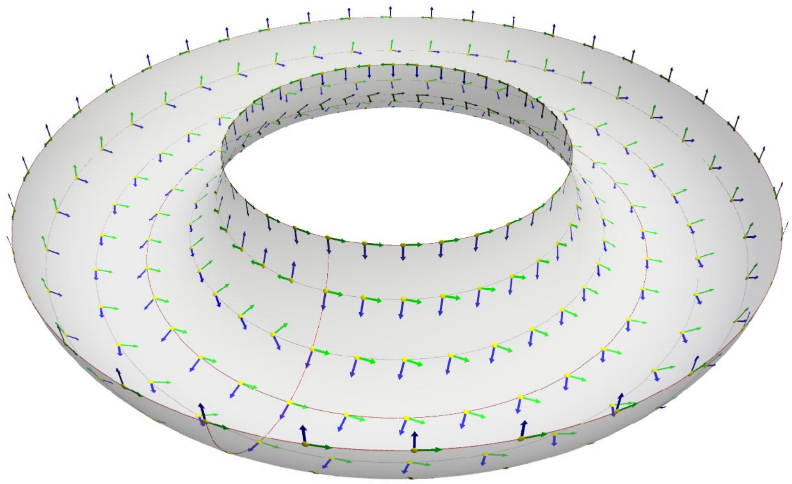


Putting all of this together, here's how a whole bunch of frames rotate while being parallel transported widdershins along lines of latitude, starting at the red longitude line.



When creating these images, I was surprised by how quickly the values of $2\pi \sin v$ change near $v = 0$. Then I remembered that $\frac{d \sin v}{dv} = \cos v$, which has extremes at integer multiples of $\pi$. So the rate of change of the effect of parallel transport along lines of latitude is most extreme at the outer and inner equators.

Finally, parallel transport on the bottom half of the torus is the same except for direction of rotation, since $\sin v$ is negative there.



Students of differential geometry may have noticed that $\sin v\, du$ is also the $\omega_{12}$ of the sphere. The difference is that for the sphere, the domain of $v$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, while for the torus it is $[-\pi, \pi]$. Parallel transport on the outer half of the torus mirrors parallel transport on the sphere.

While we're at it, let's recompute the Gaussian curvature from $E$ and $G$:

$$K = \frac{-1}{\sqrt{EG}}\left\{\left(\frac{(\sqrt{G}\,)_u}{\sqrt{E}}\right)_u + \left(\frac{(\sqrt{E}\,)_v}{\sqrt{G}}\right)_v\right\} = \frac{-1}{a(c+a\cos v)}\left(\frac{-a\sin v}{a}\right)_v = \frac{(\sin v)_v}{a(c+a\cos v)} = \frac{\cos v}{a(c+a\cos v)} \quad \checkmark$$

This agrees with the value for the Gaussian curvature we computed from the shape operator, but unlike that calculation this one doesn't require a normal to the surface. Thus, if we lived on the torus, we could compute our space's Gaussian curvature directly from measurements made within our space, without assuming the existence of an embedding space. That's the beauty of differential geometry.